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Review Article

Non-reflecting Boundary Conditions

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Past and recent research on the use of non-reflecting boundary conditions in the numerical solution of wave problems is reviewed. Local and nonlocal boundary conditions are discussed, as well as special procedures which involve artificial boundaries. Various problems from different disciplines of applied mathematics and engineering are considered in a uniform manner. Future research directions are addressed.

1. INTRODUCTION

The numerical solution of wave problems is a challenge common to many branches of engineering and applied mathematics. One aspect which must be considered when solving boundary value problems numerically, and which has both theoretical and computational importance, is the treatment of boundary conditions. In many cases the boundary under consideration is the actual boundary of the spatial domain. The choice of a good physical boundary condition for various problems and the way to combine this condition with the numerical scheme employed in the interior is an important subject of research. However, the present paper is concerned with another important type of boundary conditions, namely *artificial* boundary conditions. For a general discussion on both actual and artificial boundary conditions see the review paper by Turkel [1].

The need for artificial boundary conditions arises when the spatial domain of the problem at hand is *unbounded*. In that case, a numerical treatment usually requires the introduction of an *artificial boundary* $\mathscr{B}$, in order to make the computational domain finite. Now one has to impose some boundary condition on $\mathscr{B}$. The appropriate boundary condition to be used on $\mathscr{B}$ for various wave problems is the subject of this survey.

Consider, for example, the setup shown in Fig. 1. Suppose the two-dimensional or three-dimensional reduced wave equation

$$\nabla^2 u + k^2 u = 0 \tag{1}$$

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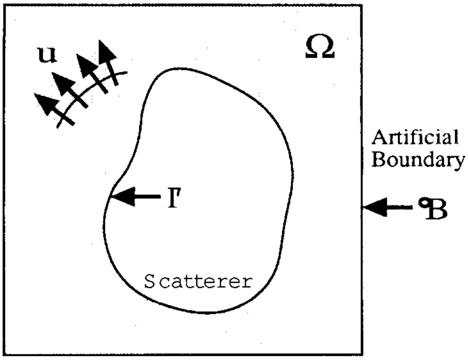

FIG. 1. The geometry of a typical exterior scattering problem.

is to be solved outside a given scatterer with a boundary Γ. Here u is the scattered field and k is the wave number. A boundary condition, which is derived from the incident wave, is given on Γ. In addition, a so-called *radiation condition* at infinity is given which states that the waves there are outgoing. An appropriate and very well-known radiation condition was given by Sommerfeld [2]:

$$\lim_{r \to \infty} r^{(d-1)/2}(u_r - iku) = 0. \tag{2}$$

Here r is the radial coordinate, $u_r = \partial u/\partial r$, and d is the spatial dimension.

In order to solve the problem numerically in a finite computational domain, the artificial boundary $\mathscr{B}$ is introduced. The computational domain, denoted Ω, is bounded internally by Γ and externally by $\mathscr{B}$ (see Fig. 1). In order for the statement of the problem in Ω to be complete, one needs to impose a boundary condition on $\mathscr{B}$. This boundary condition must have the property that waves hitting $\mathscr{B}$ from inside the computational domain Ω are transmitted through $\mathscr{B}$ without any reflection. At first sight the construction of such a condition seems to be a simple matter, but this apparent simplicity is deceptive. The obvious choice is to use a boundary condition on $\mathscr{B}$ which has the same form as (2), namely

$$u_r - iku = 0 \qquad \text{on } \mathscr{B}. \tag{3}$$

However, it is now a well-known fact that this boundary condition may produce large *spurious reflection* of waves from $\mathscr{B}$. In other words, it may lead to large errors in the computed solution.

To demonstrate this fact, consider the problem of solving Eq. (1) in the infinite plane exterior to a circle Γ of radius a. We choose $k = 1$, $a = 4$, and we prescribe $u = 1$ on $r = a$. If we plotted the contour lines of the real and imaginary parts of the *exact* solution, we would obviously get concentric circles around Γ. Now we solve the problem numerically by the finite element method. We first introduce a circular artificial boundary $\mathscr{B}$ of radius 8, to make the computational domain finite.

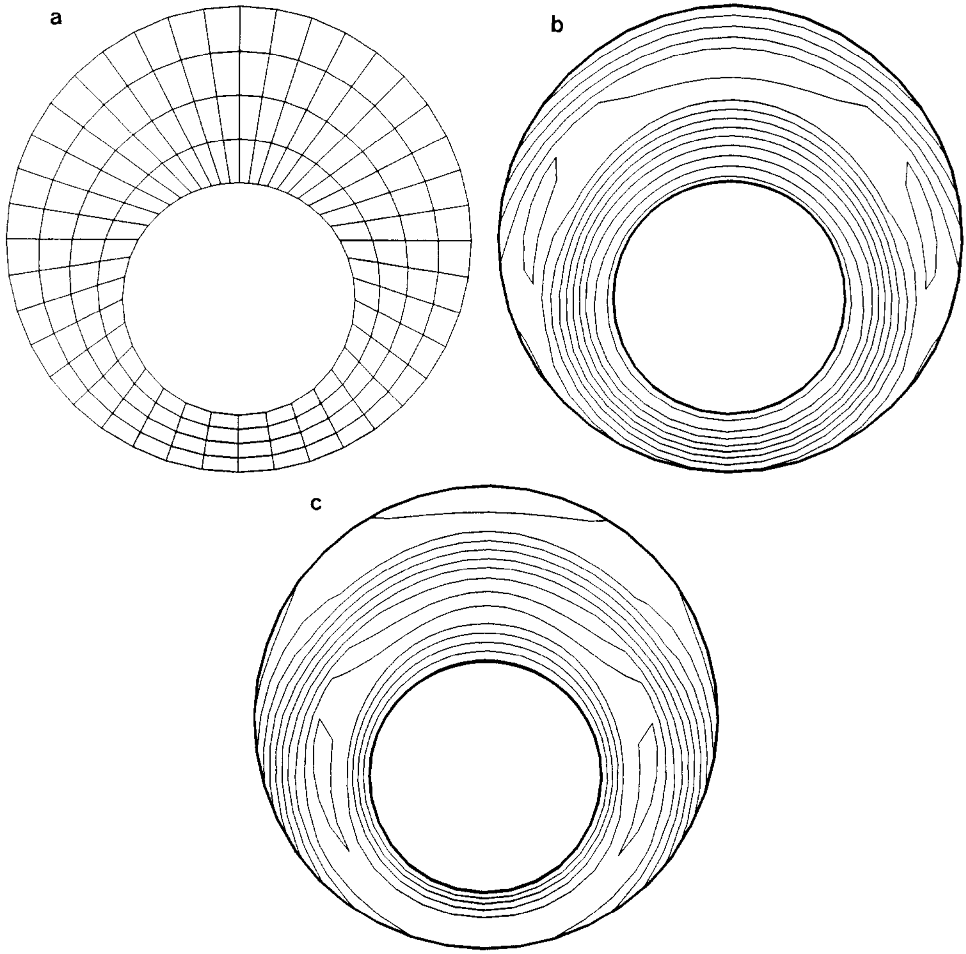

FIG. 2. Demonstration of spurious reflection of waves from an artificial boundary $\mathscr{B}$. A Sommerfeld-like boundary condition is used on $\mathscr{B}$: (a) the finite element mesh; (b) contour plot of the real part of the numerical solution; (c) contour plot of the imaginary part of the numerical solution.

Figure 2a describes the mesh, composed of bilinear quadrilateral elements. Note that $\mathscr{B}$ is chosen not to be concentric to Γ (the eccentricity is 2). On $\mathscr{B}$, the Sommerfeld-like boundary condition (3) is imposed. Figures 2b and c are the contour plots of the real and imaginary parts of the finite element solution, respectively. The spurious reflections are prominent. The accuracy of the numerical result (not given here) is poor too. If we chose Γ and $\mathscr{B}$ to be concentric, the spurious reflections would not be visible in the contour plots, but the accuracy of the results would be just as bad.

As Roe [3] writes in his paper, "A recurring frustration in Computational Fluid Dynamics is the apparent difficulty of giving numerical expression to very simple statements." Indeed, this applies to other areas of applied mathematics and applied mechanics as well. The statement that we want to express here is that the boundary $\mathscr{B}$ is transparent to waves of any kind.

There has been a considerable amount of work to devise boundary conditions that reduce the amount of spurious reflection. Results of this effort can be found in the literature related to various fields, such as acoustics, gas dynamics, hydrodynamics, electrical engineering, civil engineering, geophysics, meteorology, environmental science, and plasma physics. The geometry and governing equations considered in these fields are sometimes different, but the goals and techniques are similar. Figure 3 shows a typical setup for problems in geophysics on one hand, and for problems in meteorology on the other. These two types of problems are set in two complementary half-spaces, and the earth surface is a common boundary. In the figure, the subscripts G and M stand for "geophysics" and "meteorology," respectively. Traditionally, both artificial boundaries $\mathscr{B}_M$ and $\mathscr{B}_G$ are composed of plane surfaces.

We have found that in many cases the methods, experience and conclusions regarding the use of non-reflecting boundary conditions were not fully shared by researchers of remote scientific areas. One symptom of this fact is that researchers of a certain field tended to reference publications related to the same field. Another symptom is the variety of names that were given to non-reflecting boundary conditions. They have been also called radiating, absorbing, silent, transmitting, transparent, open, free-space, and one-way boundary conditions. In this light, the purpose of the present paper is to review the subject in a uniform manner, while referring to the literature of all the pertinent fields. We shall use the term *non-reflecting boundary condition* (NRBC) throughout this paper.

We may have left the impression that a well-designed NRBC should let waves that come from inside the computational domain Ω go out, while preventing the entrance of waves that approach $\mathscr{B}$ from the exterior. This indeed is an assumption commonly made in the development of NRBCs. However, it should be noted that

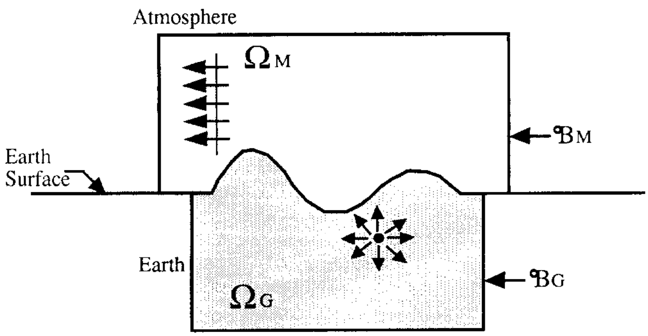

FIG. 3. A typical setup for problems in geophysics and meteorology. Problems in geophysics are set in the lower half-space, whereas problems in meteorology are set in the upper one.

there are some physical situations in which waves actually enter into Ω from outside. One obvious example is the case in which an obstacle or a hole, or even some inhomogeneity, is present in the exterior domain. A second example (for which I thank one of the referees) is as follows. Consider a nonlinear hyperbolic conservation law, and suppose that two shock waves leave Ω and collide outside Ω. This self-interaction in the solution may send waves back into the computational domain. Thompson [51], Hedstrom [50], Hagstrom and Hariharan [56], and Karni [85] treat situations like this. Another example is given by Ferm and Gustafsson [115].

One may argue that for a boundary condition which is devised to deal with such cases the adjective "open" or "free-space" may be more appropriate than "non-reflecting." However, by a NRBC we mean a condition which does not give rise to *spurious* reflections (as opposed to true physical reflections), and therefore the special cases mentioned above are included as well.

The use of an artificial boundary in a numerical calculation enables one to eliminate the exterior domain from the computation. Therefore, it is clear that the boundary condition is solely responsible for the correct representation of this eliminated domain and the physical phenomena that occur therein. In fact, some methods of constructing NRBCs are based on the properties of the exterior solution or the far-field solution, and others assume (implicitly or explicitly) that the exterior domain possesses some "regularity." For instance, in problems of elasticity it is common to assume that the exterior medium is isotropic and homogeneous, that no loads act on it, and that it is linear both geometrically and materially. On the other hand, inside Ω the only limitations are those dictated by the capabilities of the numerical scheme employed.

In devising a new NRBC one has in mind at least some of the following goals:

1. The problem in Ω together with the boundary condition on $\mathcal{B}$ is mathematically *well posed.*

2. The problem in Ω together with the boundary condition on $\mathcal{B}$ is a *good approximation of the original problem* in the infinite domain.

3. The boundary condition on $\mathcal{B}$ is highly *compatible with the numerical scheme* used in Ω.

4. The numerical method employed together with the boundary condition used on $\mathcal{B}$ must result in a *stable numerical scheme.*

5. The amount of *spurious reflection* generated by the boundary condition on $\mathcal{B}$ is small.

6. The use of the boundary condition on $\mathcal{B}$ does not involve a large *computational effort.*

7. In time-dependent schemes where only the steady state solution is sought, the numerical scheme should *reach the steady state rapidly.*

In the list above, properties 1 and 2 have to do with the continuous problem, prior to the introduction of the numerical scheme, whereas properties 3–7 deal with the approximate discrete problem. Usually, the combination of properties 2 and 3 implies property 5, although it is easier to check property 5 directly. Also, roughly speaking, the satisfaction to a high degree of properties 1–4 usually implies the convergence of the numerical scheme. Of course, one has to prove that the method ensures convergence, but this is more reasily done after the NRBC has been introduced and not in the process of designing it.

The combination of properties 5 and 6, namely the reduction of spurious reflections to a minimum in an efficient way, has been the main object of most researchers. Most NRBCs perform well if the artificial boundary $\mathscr{B}$ is set far away from all sources or scatterers. In fact, in solving time-dependent problems one can set $\mathscr{B}$ sufficiently far away so that no waves would reach it in the time interval in which the solution is sought. However, this would result in a large computational domain Ω, and is therefore inefficient. Hence, the NRBC has to perform well even when set quite close to the source or scatterer. In addition, it must not be in itself so complicated as to require a large computational effort.

Following is the outline of the succeeding sections. First we survey the research that has been done on NRBCs which are local in space and in time. In Section 2 we consider problems governed by the scalar wave equation and by the reduced wave equation. In Section 3 we discuss local NRBCs in the context of gas dynamics, hydrodynamics, meteorology, elasticity, and electromagnetism. In Section 4 we refer to special procedures that involve an artificial boundary without the direct use of a NRBC. In Section 5 we discuss boundary conditions which are nonlocal in space or in time or both. This is a much smaller group than that of local boundary conditions, but the nonlocal conditions are more accurate and some of them are as efficient as the local conditions. Finally, we mention some future research directions in Section 6.

2. Local Boundary Conditions: Scalar Wave Equation

In this section we consider NRBCs which are *local* in both space and time, for use in problems governed by the scalar wave equation

$$u_{tt} = c^2 \nabla^2 u. \tag{4}$$

Here t is time and c is the wave speed. We are also interested in the special case of time-harmonic waves, i.e., waves that have the form

$$u(\mathbf{x}, t) = \tilde{u}(\mathbf{x})\, e^{i\omega t}. \tag{5}$$

Here ω is the wave frequency. Substituting Eq. (5) in (4), we obtain the reduced wave equation (1) for $\tilde{u}$, where $k = \omega/c$. NRBCs corresponding to Eq. (4) and to Eq. (1) have been considered mainly for applications in acoustics.

The radiation condition at infinity analogous to (2) for the time-dependent equation (4) is

$$\lim_{\substack{r \to \infty \\ r + ct = \text{const.}}} r^{(d-1)/2} \left(u_r + \frac{1}{c} u_t \right) = 0. \tag{6}$$

We see that $(1/c)\, u_t$ in (6) is replaced by $-iku$ in (2). Similarly, any NRBC which was devised for the time-dependent case, can be adapted to the time-harmonic case by simply replacing every occurrence of the operator $\partial/\partial t$ by $-i\omega$.

We first consider the one-dimensional case. We note that in this case the radiation condition (6), which reduces to

$$u_t + cu_x = 0, \tag{7}$$

is exact not only at infinity but also at any finite point. If we think of the domain as representing a semi-infinite vibrating rod, the physical interpretation of this statement is that the rod can be made finite by truncating it at any point and replacing the eliminated part with a dashpot of strength $1/c$. Halpern [4] considered various finite difference approximations of condition (7) and discussed their stability. Foreman [5] examined the accuracy of an explicit finite difference scheme versus that of a finite element scheme with Crank–Nicolson time stepping, for the one-dimensional linearized shallow water equations, with an exact boundary condition analogous to (7).

Clearly, the real challenges lie in two- and three-dimensional problems. Perhaps the most referenced work on NRBCs is the one by Engquist and Majda, reported in [6, 7]. They have developed a special technique, based on the theory of pseudodifferential operators (see, e.g., [8]), to obtain a sequence of local approximate boundary conditions of increasing order. In order to understand this technique, we consider for example the two-dimensional wave equation in cartesian coordinates,

$$\frac{1}{c^2} u_{tt} = u_{xx} + u_{yy}. \tag{8}$$

The substitution of the exponential solution

$$u = e^{i(-\omega t + k_1 x + k_2 y)} \tag{9}$$

in (8), or alternatively the employment of the Fourier transform, gives us the dispersion relation

$$k^2 = \frac{\omega^2}{c^2} = k_1^2 + k_2^2. \tag{10}$$

Consider now a straight segment of the artificial boundary $\mathscr{B}$ with an outward normal in the positive x direction. Denoting $s = k_2/k$ ($|s| \leqslant 1$), we have from (10),

$$k_1 = \pm k \sqrt{1 - s^2} \qquad \text{on } \mathscr{B}. \tag{11}$$

The plus and minus sign in (11) represent outgoing and incoming plane waves, respectively. In order to obtain an equation on $\mathscr{B}$ which admits only outgoing waves, the branch corresponding to the plus sign is chosen.

Now, consider (11) as a one-dimensional dispersion relation of some equation

$$Pu = 0 \qquad \text{on } \mathscr{B}. \tag{12}$$

This equation, obtained by applying the inverse Fourier transform to (11), is an exact relation on $\mathscr{B}$. Since $k_1(s)$ in (11) is an irrational function of s, the operator P in (12) is not a differential operator, but rather a pseudodifferential operator. It is nonlocal in both space and time, and is not practical for computations. Engquist and Majda's technique is based on approximating the nonlocal pseudodifferential operator P by a local differential operator E. This is done by approximating the irrational function $\sqrt{1 - s^2}$ in (11) by a rational function. Using rational approximations of increasing accuracy, Engquist and Majda obtain local boundary conditions $E_m u = 0$ on $\mathscr{B}$ of increasing order.

In [6], Engquist and Majda first derive a sequence of NRBCs for Eq. (4) in two dimensions in rectangular coordinates. The first two conditions are:

$$\begin{aligned} \bar{E}_1 u &= \left(\frac{\partial}{\partial x} - \frac{1}{c} \frac{\partial}{\partial t} \right) u = 0 \\ \bar{E}_2 u &= \left(\frac{1}{c} \frac{\partial^2}{\partial x \, \partial t} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} + \frac{1}{2} \frac{\partial^2}{\partial y^2} \right) u = 0. \end{aligned} \tag{13}$$

The boundary condition $\bar{E}_1$ is perfectly absorbing for plane waves at normal incidence. Next, the authors generalize these boundary conditions for the case in which the computational domain is inhomogeneous (namely the wave equation has variable coefficients there) and the support of this inhomogeneity reaches the boundary $\mathscr{B}$. The first of these generalized NRBCs contains only first-order derivatives, like $\bar{E}_1$, but the second contains third-order derivatives, unlike $\bar{E}_2$. For the homogeneous case the latter condition reduces to the time derivative of $\bar{E}_2$. These two NRBCs are used and compared in several numerical examples.

In [6] NRBCs in polar coordinates are derived as well, where $\mathscr{B}$ is a circle of radius R. In general, there are three ways to construct NRBCs in polar coordinates. First, boundary conditions may be specially developed in a polar system from the start. Second, that NRBCs that were designed for rectangular coordinates can be used in polar coordinates by replacing x and y by r and $R\theta$. Naturally, NRBCs used without caution in this way may lead to inaccuracies in the solution. The third way, which is adopted by Engquist and Majda, is to regard the wave equation in polar coordinates as a special case of the wave equation in rectangular coordinates with variable coefficients. Thus, the first two NRBCs obtained in [6] for the inhomogeneous case reduce to:

$$
\begin{aligned}
E_1 u &= \left(\frac{\partial}{\partial r} + \frac{1}{c}\frac{\partial}{\partial t} + \frac{1}{2R}\right) u = 0 \\
E_2 u &= \left(\frac{1}{c^2}\frac{\partial^3}{\partial r\,\partial t^2} + \frac{1}{c^3}\frac{\partial^3}{\partial t^3} - \frac{1}{2R^2 c}\frac{\partial^3}{\partial t\,\partial^2\theta} + \frac{1}{2Rc^2}\frac{\partial^2}{\partial t^2} + \frac{1}{2R^3}\frac{\partial^2}{\partial \theta^2}\right) u = 0.
\end{aligned}
\tag{14}
$$

According to Claerbout [9], rational approximations were used to develop *one-way wave equations*, before Engquist and Majda realized that similar ideas can be used to design NRBCs. A one-way wave equation is a differential equation which permits wave propagation in certain directions only. Such equations are used in the simulation of the geophysical migration of seismic waves [10] and in underwater acoustics calculations [11]. Lindman [18] seems to be the first to have suggested that one-way wave equations can be applied as NRBCs.

In the context of one-way wave equations, Trefethen and Halpern [12] and Halpern and Trefethen [13] considered the approximation of $T(s) = \sqrt{1-s^2}$ by a rational function $\rho(s)$ in $[-1, 1]$. Whereas Engquist and Majda used Padé approximations only, in [12, 13] several other approximations are considered as well, including Chebyshev, Chebyshev–Padé, Newman, L^2, and L^∞ approximations. In [13] the authors find the coefficients in the first few one-way wave equations (or NRBCs) in each case, and compare the various approximations in several numerical experiments. One observation from these experiments is that Padé approximants are by far the best at nearly normal incidence, and by far the worst at nearly tangent incidence.

In [12] Trefethen and Halpern identify those functions $\rho(s)$ which ensure a well-posed boundary value problem. The analysis is based on the theory of Kreiss [14] for checking well-posedness for mixed boundary value problems. One of the results is that the problem is well posed if $\rho(s)$ interpolates $T(s)$ for a sufficiently large number of points in $(-1, 1)$. A well-posedness analysis of the Engquist and Majda boundary conditions for the linearized shallow water equations, was presented by Kolakowski in [15, 16].

Halpern and Rauch [17] obtained estimates on the error made by using Engquist and Majda's boundary conditions in the continuous setup. The proofs assume that $\mathscr{B}$ is convex, smooth, and has a strictly positive curvature, so rectangles are not covered by this analysis.

For equations with variable coefficients outside Ω, Engquist and Majda [7] propose to use one of two procedures: either to compute the nonlocal theoretical boundary condition asymptotically at large frequencies and then to localize it, or to use the boundary conditions devised for the constant coefficient case while "freezing" these coefficients. The authors show in a numerical example that the two procedures give similar results.

In their work, Engquist and Majda first derived NRBCs for the continuous problem, and only then discretized the equations and boundary conditions. In [7], they also consider the possibility of first discretizing the differential equations and then deriving local boundary conditions which have good transmitting properties with respect to the difference equations. This procedure was indeed used by Lindman [18] and later by Randall [19]. They both used finite difference approximations in space and time. Engquist and Majda show that this approach is better than the continuous approach only if the solution contains high frequencies which are represented by a small number of grid points per wave length. However, this is an undesirable situation, because in this case the grid is too crude for the problem at hand. In all other cases the continuous approach is preferable.

A very detailed report on the application of the Engquist and Majda boundary conditions for the two-dimensional reduced wave equation, was given by Behrendt [20]. A hybrid finite element formulation was used in Ω. Both infinite and semi-infinite domains were considered. Hariharan and Bayliss [21] implemented the three-dimensional version of the boundary condition E_1 in (14), and solved the problem of sound radiation into the atmosphere from a cylindrical pipe.

Wagatha [22] started from the theoretical nonlocal condition of Engquist and Majda, but considered local approximations of this condition that depend on a free parameter β. The local boundary conditions thus obtained reduce to Engquist and Majda's local conditions for certain choices of β. However, Wagatha shows that smaller spurious reflections can be obtained by choosing β in an optimal way. The main improvement is obtained in cases where the angle of incidence is not close to being normal. Another set of local boundary conditions with adjustable free parameters was proposed by Clayton and Engquist [23]. The well-posedness of these conditions for certain choices of the parameters was checked by Howell and Trefethen [24]. Their analysis was accompanied by some illustrative numerical examples.

Reynolds [25] used a technique similar to that of Engquist and Majda, although less rigorous, and obtained a local boundary condition in cartesian coordinates which is a modified version of the second boundary condition in [7]. A scheme for using the proposed boundary condition for elastodynamics was also discussed.

Bayliss and Turkel [26] obtained a sequence of NRBCs for the wave equation (4) with axial symmetry and with spherical symmetry. Their boundary conditions are based on an asymptotic expansion of the solution at large distances. In three dimensions, their mth boundary condition is

$$B_m u = \left(\prod_{j=1}^{m} \left(\frac{1}{c} \frac{\partial}{\partial t} + \frac{\partial}{\partial r} + \frac{2j-1}{R} \right) \right) u = 0 \qquad \text{on } \mathscr{B}. \tag{15}$$

The authors prove that as $\mathscr{B}$ approaches infinity, the distance (in the H^m norm) between the solution of the wave equation that satisfies $B_m u = 0$ on $\mathscr{B}$ and the solution of the original problem in the unbounded domain is $O(R^{-m-1/2})$, where R is the radius of $\mathscr{B}$. The numerical example that is given demonstrates that B_1 is better than the Sommerfeld condition, and that B_2 is still better. For the reduced wave equation in two dimensions an analogous sequence of boundary conditions is obtained:

$$B_m u = \left(\prod_{j=1}^{m} \left(-ik + \frac{\partial}{\partial r} + \frac{4j-3}{2R} \right) \right) u = 0 \qquad \text{on } \mathscr{B}. \tag{16}$$

A finite difference scheme is used in the computational domain.

In a later paper, Bayliss and Turkel [27] extend these ideas to the linearized compressible Euler equations in two and three dimensions, and to wave guides. A wave guide geometry is described in Fig. 4. Here, the artificial boundary $\mathscr{B}$ is the vertical surface that divides the original domain into the finite "irregular" domain Ω (the computational domain), and the semi-infinite "regular domain" (the domain eliminated). The latter domain is sometimes called a "constant tail." Figures 1, 3, and 4 describe the main geometrical configurations that are considered in scattering problems.

Bayliss and Turkel also consider in [27] the effectiveness of their boundary conditions in the case where one desires a rapid path to the steady state solution using a time-dependent scheme. In [28], Bayliss *et al.* use the same boundary conditions with a finite element method in Ω. From the conditions (16), only the first ($m=1$) is compatible with the standard C^0 finite element formulation. To make the $m=2$ condition compatible as well, the second derivative in r is eliminated by using the differential equation (1) itself.

Feng [29] considers the reduced wave equation (1) in two dimensions, and obtains a sequence of local NRBCs. He does that by first deriving an exact nonlocal integral relation on the boundary $\mathscr{B}$ using the appropriate Green's function. Then this nonlocal relation is localized by using an asymptotic approximation valid at large distances. For a circular artificial boundary of radius R, Feng's first four conditions are

$$\begin{aligned} F_0 u &= -u_r + iku = 0 \\ F_1 u &= -u_r - \left(-ik + \frac{1}{2R} \right) u = 0 \\ F_2 u &= -u_r - \left(-ik + \frac{1}{2R} - \frac{i}{8kR^2} \right) u + \frac{i}{2kR^2} \frac{\partial^2 u}{\partial \theta^2} = 0 \\ F_3 u &= -u_r - \left(-ik + \frac{1}{2R} - \frac{i}{8kR^2} - \frac{1}{8k^2R^3} \right) u \\ &\quad + \left(\frac{i}{2kR^2} + \frac{1}{2k^2R^3} \right) \frac{\partial^2 u}{\partial \theta^2} = 0. \end{aligned} \tag{17}$$

In the computational domain Ω, Feng proposes to use the finite element method. This method is highly compatible with the boundary conditions (17), but higher-

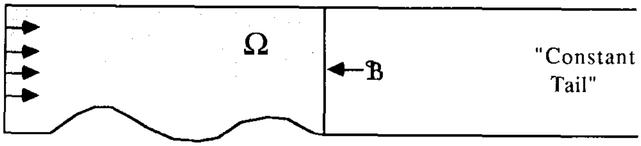

FIG. 4. A typical geometry of a wave-guide or a duct problem. The computational domain Ω may have complicated geometry, but the domain outside Ω is usually regarded as a "constant tail."

order boundary conditions (i.e., $F_m u = 0$ for $m \geq 4$) cannot be used with the standard finite element formulation.

Higdon [30] considers the two-dimensional wave equation in a rectangular computational domain Ω. He first approximates (4) by finite differences in both space and time, and then presents some discrete boundary conditions to be used on $\mathscr{B}$. He shows that the continuous counterparts of these discrete boundary conditions are equivalent to boundary conditions of the form

$$H_m u = \left(\prod_{j=1}^{m} \left((\cos \alpha_j) \frac{\partial}{\partial t} - c \frac{\partial}{\partial x} \right) \right) u = 0. \tag{18}$$

The boundary condition (18) is perfectly absorbing for a plane wave hitting the boundary $\mathscr{B}$ at one of the angles $\pm \alpha_j$ for $j = 1, ..., m$. Although in most applications the direction in which the waves approach the boundary is unknown a priori, Higdon's numerical experiments suggests that the amount of spurious reflection is not very sensitive to the choice of the α_j. Also, a reasonably small value of m leads to a boundary condition which absorbs waves quite well for a wide range of angles of incidence. In [31], Higdon performs a detailed analysis of various finite difference approximations of (18). The stability and reflection properties of these conditions are examined.

In an earlier work, Keys [32] obtained the same boundary conditions, and nicely compared them with those of Engquist and Majda [6], Reynolds [25], and Lysmer and Kuhlemeyer [33] (the latter in the context of elastodynamics).

In [30], Higdon proves a very interesting theorem, which implies that several NRBCs that have been proposed previously are special cases of (18). The theorem states that if a NRBC is based on a *symmetric rational approximation* to the dispersion relation corresponding to outgoing waves, then it is either (a) equivalent to (18) for a suitable choice of m and the angles α_j, or (b) unstable, or (c) not optimal, in that the coefficients in the NRBC can be modified so as to reduce the amount of the spurious reflection, measured by the reflection coefficient of each Fourier mode. In other words, any stable NRBC that is derived by using a symmetric rational approximation and that cannot be improved by a simple modification of its coefficients, is characterized completely by its angles of perfect absorption. The examples given by Higdon to demonstrate this theorem include the NRBCs of Engquist and Majda [6], Wagatha [22], and Trefethen and Halpern [12].

Higdon's theorem seems to imply that there is not much value in using rational approximations to derive NRBCs for the scalar wave equation, since equivalent boundary conditions can be obtained in a simpler way by using (18). The fact that (18) requires the choice of the parameters α_j may seem as a disadvantage, but one may argue that those NRBCs which are special cases of (18) implicitly make this kind of choice for the analyst. The combination of several useful local NRBCs is also beneficial from a programmer's point of view. However, it must be remembered that rational approximations of dispersion relations are of great importance in the derivation of one-way wave equation, as *interior* differential equations. This fact is emphasized in Trefethen and Halpern's work [12].

Kriegsmann and Morawetz [34] obtain a NRBC for two-dimensional time-harmonic waves. They start from the time-dependent equation

$$(Qu)_t = \nabla^2 u + k^2 u, \tag{19}$$

where Q is an operator chosen so that the solutions of (19) which have the form of a plane wave plus an outgoing scattered wave, approach a steady state as time goes to infinity. These solutions are shown to satisfy the reduced wave equation (1). For a circular boundary $\mathscr{B}$ of radius R, the NRBC is

$$2kR^2(i - 1/kR)\left(u_r + \frac{1}{c}u_t\right) = \left(\frac{\partial^2}{\partial\theta^2} + \frac{1}{4}\right)u \qquad \text{on } \mathscr{B}. \tag{20}$$

It is obtained from the asymptotic solution of (19) for large distances. A finite difference scheme is used in Ω, and steady state is reached after marching a sufficient number of time steps. A number of numerical examples are presented, including one related to plasma physics. A similar technique was applied by Kriegsmann [35] for two-dimensional wave guides.

Engquist and Halpern [36] consider the dispersive wave equation, and propose to use a NRBC of the form

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial n} + Ku = 0 \qquad \text{on } \mathscr{B}, \tag{21}$$

where K is the time-independent operator appearing in the corresponding boundary condition $\partial u/\partial n + Ku = 0$ for the steady state. The operator K may be either non-local or local in space. The authors present some one-dimensional numerical examples, using finite element discretization. They prove well-posedness and the convergence to steady state as $t \to \infty$. They also discuss the extension to hyperbolic systems.

Problems of fluid–structure interaction are considered in various situations. Underwater acoustic waves encountering a submerged body is one example. Water waves hitting an offshore structure is another. The fluid medium is almost always considered infinite in these cases, whereas the solid medium may be considered finite or infinite. In seismology, one is often interested in the interaction between water waves of seismic origin and the elastic waves inside the walls of a reservoir or a dam in a river. In the latter case the geometry is that of a wave guide, as in Fig. 4, where the surface of the dam is the left vertical boundary. The ground is usually assumed to move in a horizontal harmonic motion. The part of the ground floor which is the boundary of Ω may have complicated geometry (due to rocks, hills, etc.), but the part of it outside $\mathscr{B}$ is a "constant tail," namely it is assumed to be uniform.

Sharan [37] considers this configuration in two dimensions. The governing equation in the fluid region is the reduced wave equation (1), where u here is the excessive pressure, while in the solid the equations of linear time-harmonic elastodynamics govern. The finite element method is used in both regions. Sharan obtains the solution of (1) as a Fourier series in the coordinate normal to the water surface, and uses the leading term of this series to derive a NRBC. This condition is shown to be valid only for sufficiently small wave numbers, and to perform well if $\mathscr{B}$ is far enough from the solid. A similar procedure was applied to other problems of fluid–solid interaction by Zienkiewicz and Newton [38], Bando *et al.* [39], and Sharan [40].

3. Local Boundary Conditions: Other Types of Waves

We move to consider NRBCs for problems in gas dynamics, hydrodynamics, and meteorology. Pearson [41] used a Sommerfeld-type boundary condition

$$u_t + cu_n = 0 \qquad \text{on } \mathscr{B}, \tag{22}$$

in a finite difference scheme for the two-dimensional incompressible Navier–Stokes equations. In (22), u_n is the normal derivative on $\mathscr{B}$. Instead of giving the phase velocity c a constant value, the author evaluated the propagation velocity on $\mathscr{B}$ from a linearized dispersion relation. Orlanski [42] also used (22) as a boundary condition, but calculated the propagation velocity on each boundary grid point from data on the neighboring grid points. Additional variations of these ideas, for different problems in meteorology, were proposed by Raymond and Kuo [43], Miller and Thorpe [44], Klemp and Lilly [45], Carpenter [46], and Wurtele *et al.* [47]. They all used various finite difference approximations in Ω.

Engquist and Majda [48] devised absorbing boundary conditions for the transonic small disturbance equation of unsteady flows, using their pseudo-differential technique. Two difficulties appear in this case. First, the governing equation is nonlinear, and second, the wave propagation speed is arbitrarily large on $\mathscr{B}$. The authors circumvent the first difficulty by "freezing" the nonlinearity, and constructing the NRBC as if the equation was linear. The second difficulty requires using approximations to the square root function which are somewhat different than those used in [6, 7]. Jiang and Wong [49] used a slightly different formulation, in which they sought rational approximations of the absolute value function rather than approximations of the square root function.

Hedstrom [50] considered a certain class of nonlinear hyperbolic systems in one spatial dimension, and obtained a NRBC which performs well if there are no strong outgoing shocks. A finite difference scheme was used in Ω. Thompson [51] extended Hedstrom's ideas to the two-dimensional case. He considered the problem of the homologous expansion of an adiabatic gas, where some interaction exists between the computational domain and the exterior. In a later paper, Thompson [52] discusses this kind of problem in more detail. Rudy and Strikwerda [53] included a free parameter in Hedstrom's boundary condition and found an optimal value of this parameter. The optimization was based on the criterion that the solution must reach steady state as quickly as possible. In [54], the same authors apply their scheme to a series of test problems. Wilson [55] obtained, in a similar context, a discrete boundary condition which is nonlocal in time, and then localized it. He also showed how to adapt this boundary condition to multi-dimensional problems.

We already mentioned Bayliss and Turkel's work [27] for the linearized Euler equations. Hagstrom and Hariharan [56] also used an asymptotic solution of the far field equations to obtain a NRBC for the nonlinear Euler equations in the spherically symmetric case. They compared their condition to Thompson's [51] in a number of numerical examples.

In the context of hydrodynamics, Kim *et al.* [57] considered weakly dispersive tsunami waves, governed by the two-dimensional Boussinesq equation for the water level anomaly. This is a nonlinear fourth-order partial differential equation. Using some order-of-magnitude arguments for the five terms in this equation, the authors derive a local NRBC on the boundary $\mathscr{B}$ of a rectangular computational domain Ω. In Ω a finite difference approximation was used with a ray-following scheme. In a number of numerical experiments the proposed boundary condition was shown to be much better than the standard Sommerfeld condition, although spurious reflections remained in some cases.

Next we move to consider elastic waves. NRBCs for elastic waves have been studied mainly in the context of geophysics. Most researchers have studied the geometry described in the lower half of Fig. 3, in two dimensions. In what follows we denote the coordinates normal and tangent to $\mathscr{B}$ by x_1 and x_2, respectively. We also let u_1 and u_2 denote the displacements and T_1 and T_2 the tractions in these directions.

The oldest NRBC for elastic waves was probably proposed by Lysmer and Kuhlemeyer [33]. Their boundary condition is often referred to as the classical viscous boundary condition. It has the form

$$\begin{aligned} a\rho c_{\mathrm{L}} \frac{\partial u_1}{\partial t} &= T_1, \\ b\rho c_{\mathrm{T}} \frac{\partial u_2}{\partial t} &= T_2. \end{aligned} \tag{23}$$

Here ρ is the mass density, c_{L} is the longitudinal wave (P wave) speed, c_{T} is the transverse wave (S wave) speed, and a and b are dimensionless parameters. The parameters a and b were chosen to minimize the reflected energy for an incident plane wave hitting the boundary $\mathscr{B}$ at a given angle of incidence. The choice of a and b was considered separately for an incident longitudinal wave, for an incident transverse wave, and for a surface wave. It was suggested that $a = b = 1$ is a good choice in general. In the computational domain Ω a finite element scheme was used.

Lysmer and Kuhlemeyer's work has inspired several other researchers to examine NRBCs for elastic waves. Their own boundary condition (23) has been found to yield large spurious reflections in certain situations, for example when the incident wave hits the boundary $\mathscr{B}$ at a sharp angle. For a discussion on the errors made by using the NRBC (23), see Castellani [58].

White *et al.* [59] used the Lysmer and Kuhlemeyer boundary condition with a certain choice of the parameters a and b. These parameters are determined by first discretizing the domain using finite elements, and then deriving linear relations between velocities and stresses on $\mathscr{B}$ from the finite element model. The authors showed that the amount of spurious reflections is thus smaller compared with the reflections obtained by using $a = b = 1$.

Clayton and Engquist [60] used the Engquist and Majda pseudodifferential technique to obtain NRBCs for elastic waves. In [60] the emphasis is on the presentation of some numerical experiments with these boundary conditions, whereas a detailed derivation and analysis were given later by Engquist and Majda [7]. The simplest of the proposed local conditions is

$$\begin{aligned} \left(\frac{\partial}{\partial t} - c_{\mathrm{L}} \frac{\partial}{\partial x_1}\right) u_1 = 0, \\ \left(\frac{\partial}{\partial t} - c_{\mathrm{T}} \frac{\partial}{\partial x_1}\right) u_2 = 0. \end{aligned} \tag{24}$$

This condition is perfectly absorbing for plane waves at normal incidence. Their next boundary condition involves a linear combination of the operators $\partial^2/\partial x_1\,\partial t$, $\partial^2/\partial t^2$, $\partial^2/\partial x_2\,\partial t$, and $\partial^2/\partial x_2^2$. These boundary conditions are used in a difference scheme employed in a rectangular domain. A special procedure is suggested to avoid instabilities near the corners.

In a comment on Clayton and Engquist's paper, Emerman and Stephen [61] report that they empirically found the proposed boundary conditions to be unstable when $c_{\mathrm{T}}/c_{\mathrm{L}} < 0.46$. They suggest using alternative discrete boundary conditions, which are roughly the time-derivative of the original ones. Mahrer [62] also performs some empirical tests with the boundary conditions of Clayton and Engquist [60] and Reynolds [25].

Sochacki [63] derives a local boundary condition based on the combination of two conditions, obtained for the cases in which only P plane waves and only S plane waves are present. The proposed boundary condition is perfectly absorbing at normal incidence, but is more complicated than (24). It is not tested numerically in the paper.

Scandrett *et al.* [64] obtain time-harmonic solutions of the equations of elastodynamics by solving numerically the time-dependent problem and using the limiting amplitude principle. They derive an approximate time-dependent boundary condition on $\mathscr{B}$, which is equivalent to the first-order Engquist and Majda condition. An analogous boundary condition for the three-dimensional case is also obtained. A finite difference scheme in time and space is employed.

Higdon [65] suggested using a difference scheme in Ω with a boundary condition on $\mathscr{B}$ of the form

$$\left(\prod_{j=1}^{m}\left(\beta_j^i \frac{\partial}{\partial t} - c_L \frac{\partial}{\partial x_1}\right)\right) u_i = 0; \qquad i = 1, 2. \tag{25}$$

Here the β_j^i are parameters which can be adjusted so that the boundary condition is perfectly absorbing for an incident plane wave hitting $\mathscr{B}$ at a given angle. This is a generalization of Higdon's boundary condition (18) for the scalar wave equation. In the case $m = 1$ with normal incidence, (25) reduces to the Engquist–Majda condition (24). If m is large enough, the resulting high-order boundary condition can be made perfectly absorbing for a number of chosen incidence angles and for both longitudinal and transverse waves. Surfaces waves were not treated. Higdon discussed the stability of his boundary conditions, and their behavior near corners.

Cohen and Jennings [66] consider what they call "paraxial boundary conditions," which they obtain by using some simple ad hoc approximations. For the scalar wave equation and for elastodynamics in two dimensions their boundary conditions are slightly modified versions of Clayton and Engquist's conditions [60]. However, their procedure enables them to obtain a simple NRBC for three-dimensional elastodynamics. For the two-dimensional case the authors perform a stability analysis, and obtain a map of stability regions whose parameters are the Poisson ratio and the angle of incidence. In the unstable regions, they propose to use a modified stable boundary condition. In several numerical tests, they compare their boundary conditions with those of Lysmer and Kuhlemeyer [33] and White *et al.* [59]. Their boundary condition turns out to perform only slightly better. In the computational domain, a finite element scheme is used. "Upwind" elements are employed to avoid oscillations due to the presence of the asymmetric term $u_{t,x}$ in the boundary conditions.

Bamberger *et al.* [67] considered time-dependent elastodynamics. They proposed to modify the first-order boundary condition of Cohen and Jennings in order to absorb Rayleigh surface waves as well. Their modified boundary condition involves the operator $(\partial/\partial t - c_R(\partial/\partial x_1))$, analogously to (24). Here c_R is the Rayleigh wave speed, which is the solution of a well-known transcendental equation. The authors proved that the proposed boundary condition is perfectly absorbing for P and S waves at normal incidence, as well as for Rayleigh waves. They used finite elements in the spatial domain together with a time-stepping scheme.

Robinson [68] considers time-harmonic elastic waves in two dimensions, and proposes a NRBC which involves the elastic potentials associated with the

Helmholtz decomposition. Both plane waves and cylindrical waves are considered. Barry *et al.* [69] propose a NRBC for the one-dimensional time-dependent problem of a semi-infinite inhomogeneous elastic bar. They use geometrical optics in the Laplace transform domain to derive their NRBC, then modify it to make it satisfy an energy stability criterion. A mixed finite element formulation is used in space, together with a finite difference scheme in time. The authors discuss possible extension to two dimensions, but mention that they encountered some stability difficulties.

We close this section with works on NRBCs for electromagnetic waves. Kriegsmann *et al.* [70] examine the electromagnetic waves scattered from a perfectly conducting cylinder. They devise a local NRBC, which they call the "On-Surface Radiation Condition" (OSRC), based on the far-field approximation of an exact integral relation involving the Green's function. The boundary condition is applied on the surface of the cylinder itself. For cylinders with simple cross sections the boundary condition is shown to perform well. The excellent review on radiation boundary operators for electromagnetic wave scattering, written by Moore *et al.* [71], focuses on the OSRC and lists about 10 related references, which will not be repeated here.

Mur [72] considered the two- and three-dimensional time-dependent Maxwell equations for a vacuum region. The Engquist and Majda boundary conditions were used for each component of the electric field separately. Numerical examples were given using the first two conditions. Other applications of the same NRBC were considered by Umashankar and Taflove [73]. Tajima [74] considered electromagnetic plasma simulations governed by the two-dimensional steady state Maxwell equations. A "masking algorithm" was used to derive a simple local boundary condition. The method is simpler than that of Lindman [18], but in a comparison between the two, the latter turned out to be more accurate. Blaschak and Kriegsmann [75] presented some finite difference schemes for the second- and third-order NRBCs of Halpern and Trefethen in the context of electromagnetic waves, and compared them with Higdon's schemes [31]. Their tests include a propagating pulse problem and a time-harmonic problem.

4. Special Procedures

In this section we consider special procedures for the numerical solution of wave problems in unbounded domains, that involve an artificial boundary but not the direct use of a NRBC. For a general discussion on various methods to solve the reduced wave equation (1), see Goldstein [76].

Grosch and Orszag [77] consider the use of an algebraic or an exponential mapping of the infinite domain to a finite domain. They show that this technique fails in certain situations, depending on the condition at infinity. Six cases in which the method works well are discussed. They include the one-dimensional wave equation and Burgers' equation.

Smith [78] shows how the reflections from a piecewise planar boundary $\mathscr{B}$ can be eliminated by adding together the numerical solutions of several problems, in each of which a certain combination of Dirichlet and Neumann boundary conditions is used on $\mathscr{B}$. If reflections are to be eliminated exactly on n plane surfaces, 2^n such solutions must be added together. For problems in elastodynamics, this procedure is applied separately to dilatational waves and to surface waves.

Still in this context, Cerjan *et al.* [79], Sochacki *et al.* [80], and Hanson and Petschek [81], each presented what can be termed a "filtering scheme." In this scheme, the amplitudes of the displacements are gradually reduced in a strip of nodes adjacent to $\mathscr{B}$. Thus, the solution is artificially damped in the vicinity of $\mathscr{B}$. On $\mathscr{B}$, the usual Dirichlet or Neumann boundary condition is used. A filtering scheme was also used by Kosloff and Kosloff [82] for the scalar wave equation and for the Schrödinger equation, and by Kurihara and Bender [83] for a model in weather prediction.

Israeli and Orszag [84] advocate the simultaneous use of a NRBC and a filtering scheme. They consider two types of filtering, which correspond to two types of modifications in the governing differential equation near the boundary $\mathscr{B}$. Both the scalar wave equation and the Klein–Gordon equation are considered. By means of numerical examples, the authors show that using a NRBC together with a filtering scheme yields better results than using each technique separately.

Karni [85] considers the one- and two-dimensional nonlinear Euler equations [illegible] waves in a layer near $\mathscr{B}$. In the second scheme the amplitude of the waves is reduced, as in the works mentioned previously. In the subsonic case waves may reenter Ω from the exterior, and the two schemes are designed not to harm these incoming waves. The author presents some very illustrative numerical examples of flow past aerofoils.

Bamberger *et al.* [67] consider Rayleigh surface waves in elastodynamics. They propose to use a computational domain Ω like the one illustrated in Fig. 5. On $\mathscr{B}$, the Cohen and Jennings boundary conditions [66] are used. If the "ears" of Ω are long enough, the Rayleigh waves which propagate to the right and to the left along

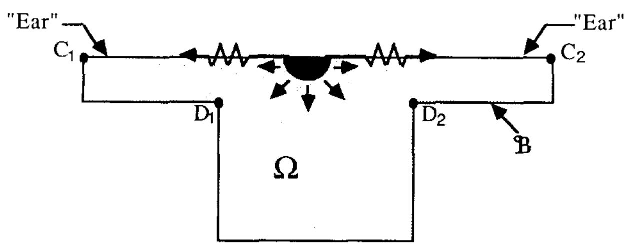

FIG. 5. The special computational domain Ω proposed by Bamberger *et al.*, in order to absorb Rayleigh surface waves.

the upper surface do not reach the points C_1 and C_2, and no spurious reflection occurs. However, the authors show that some spurious reflections from the points D_1 and D_2 of other types of waves, are present in the numerical solution.

Goldstein [86] proposes to use a finite element scheme with a non-uniform mesh, together with a simple Sommerfeld boundary condition on the artificial boundary $\mathscr{B}$. The elements gradually increase in length with increasing distances from the scatterer or source. The author shows that the mesh can be constructed so that optimal error estimates hold, and that the number of nodes is bounded in some sense. A similar technique is proposed by Day [87].

Several authors derive discrete equations on the artificial boundary, which are based on an "extrapolation formula" that involves neighboring grid points. Elvius and Sundström [88] use such a procedure for the non-linear shallow water equations, and analyze the stability of their finite difference scheme. In this scheme, a different extrapolation formula is used at even and odd time-steps. Liao and Wong [89] use a similar technique with a finite element scheme for problems in elastodynamics. Chu and Sereny [90] derive a time-dependent extrapolation formula for inviscid compressible one-dimensional problems in gas dynamics. They use a finite difference scheme, and march in time to obtain the steady state solution.

For soil–structure interaction problems in civil engineering, Underwood and Gears [91], Novak and Mitwally [92], and Lysmer and Waas [93] each employ a finite element scheme in the structure domain, and eliminate the soil domain by using some discrete relations on the interface between the two domains. Roesset and Ettouney [94] compare this last method to the one using NRBCs of the form (23) and to that of using non-uniform meshes, and conclude that it is more accurate.

Another numerical procedure that involves an artificial boundary is the coupled finite element and boundary integral method. Greenspan and Werner [95] devised the first version of this method for solving the reduced wave equation, while Mei [96] and McDonald and Wexler [97] each proposed to use another version of the method for electromagnetic waves. Later, Zienkiewicz *et al.* [98], Shaw and Falby [99], Margulies [100], Johnson and Nedelec [101], and Hsiao [102] improved the method, analyzed it, and adapted it to various problems. In this method the discrete finite element equations inside the computational domain Ω are combined with the discrete equations that result from applying the boundary element technique on $\mathscr{B}$. (See, e.g., Cruse and Rizzo [103] and Brebbia [104] on the boundary element method.) An interesting version of this method is presented by Bielak and MacCamy [105] for two-dimensional time-harmonic elastic waves in anti-plane strain state.

Still in the finite element context, we mention the use of "infinite elements" in solving infinite domain problems. See Bettess [106] and Zienkiewicz [107]. An infinite element is a semi-infinite radial strip with some nodes at infinity. Its shape functions are chosen to mimic the asymptotic behavior of the solution at infinity. In this method, some integrals over infinite domains must be calculated numerically.

5. Nonlocal Boundary Conditions

Finally, we consider NRBCs that are nonlocal in space or in time or both. Some of these boundary conditions have the potential of being more effective than any local boundary condition.

Nonlocality in time arises naturally in viscoelasticity, because the medium in this case possesses "memory." Trautenberg *et al.* [108] consider two-dimensional waves in a viscoelastic medium. First they discretize the governing equations using finite differences. The resulting discrete operator depends on the time history of the wave, as typical in viscoelastic problems. The propagation matrix is then calculated and a discrete boundary condition on $\mathscr{B}$ is found from it. The number of rows in the propagation matrix grows without limit as one steps forward in time. This implies that the discrete NRBC depends on an ever increasing amount of past data as time goes on. This might pose a great difficulty in terms of computer storage and computing time. Therefore, the propagation matrix in [108] is truncated after a fixed number of rows. Doing this amounts to limiting the memory of the NRBC. The authors report that about 20 past time-steps have to be taken into account in order to obtain accurate results.

Oddly enough, in multi-dimensional problems an exact boundary condition on an artificial boundary is inevitably nonlocal in time, even if the medium itself does not possess memory. We have seen this when discussing the Engquist and Majda theoretical boundary condition, which is nonlocal in space and time. The nonlocality is the price that one has to pay in order to eliminate an infinite spatial domain. When the problem under consideration is time-independent, a NRBC has to be spatially nonlocal in order to *exactly* represent the entire exterior domain. If the problem depends on time, then an exact condition has to represent the history of the exterior as well.

A discrete NRBC which is nonlocal in time is proposed by Wagatha [109] for various equations in meteorology. Again, in order to prevent the unlimited accumulation of past information, only a fixed number of past time-steps are taken into account. For the scalar wave equation only a small number of time-steps are needed to obtain good results; for the shallow water equations the computation time is much greater, and the method cannot compete with the efficiency of the Engquist and Majda local NRBCs.

Another nonlocal NRBC was devised by Beland and Warn [110] for two-dimensional barotropic Rossby waves in a semi-infinite channel. The Laplace transform in the semi-infinite direction was applied to the linearized far field equations. The inverse Laplace transform was calculated numerically. This lead to a boundary condition of the form

$$A \frac{\partial u}{\partial n} + Bu = \int_0^t K(t-\tau)\, u(\tau)\, d\tau \qquad \text{on } \mathscr{B}, \tag{26}$$

where A, B, and $K(t-\tau)$ are known functions. In the computational domain, a finite difference scheme was used in the direction normal to $\mathscr{B}$, and a spectral method was used in the direction parallel to it. In the numerical examples that were presented, spurious reflections were small only when the nonlinearity of the governing equation was weak.

Still in the context of meteorology, Klemp and Durran [111] considered the linear hydrostatic Boussinesq equations for gravity waves. The geometry is that shown in the upper part of Fig. 3. The authors developed a NRBC which is local in time, but nonlocal in the spatial coordinate along the boundary $\mathscr{B}$. This means that in the numerical scheme, each boundary grid point interacts not only with its neighbors, but also with all the other boundary grid points. The NRBC relates the pressure along $\mathscr{B}$ to the Fourier transform of the normal velocity at $\mathscr{B}$. In Ω, a finite difference scheme was used. The same NRBC was proposed independently by Bougeault [112].

Bennett [113] considered various problems in meteorology. By using the Laplace and Fourier transforms, he obtained a NRBC on a rectangular boundary which is nonlocal in both space and time. The paper has a negative message; the author concludes that such a boundary condition cannot be used in practice because of storage and computing time limitations.

Gustafsson and Kreiss [114] considered a hyperbolic system of equations in a waveguide. They obtained an exact nonlocal boundary condition involving the Fourier coefficients of the solution and discussed its use in a finite difference method. In [115], Ferm and Gustafsson apply this NRBC to the nonlinear Euler equations in steady state, by "freezing" the coefficients in the down-stream region. Hagstrom and Keller [116] found an exact boundary condition for a certain class of partial differential equations in cylindrical domains with a "constant tail." Their boundary condition is expressed in terms of the eigenfunctions and eigenvalues of a problem in the cross section of the cylinder. They also proved the existence of an exact boundary condition for certain nonlinear problems and gave an asymptotic expansion for it.

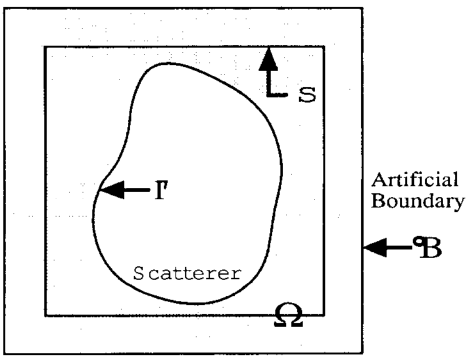

FIG. 6. The setup proposed by Ting and Miksis. Use is made of two artificial boundaries: $\mathscr{S}$ and $\mathscr{B}$. The computational domain Ω is bounded internally by the surface of the scatterer Γ, and externally by $\mathscr{B}$.

Gustafsson [117] considered the two-dimensional Euler equations of gas dynamics and first-order hyperbolic systems, in a rectangular computational domain Ω. He first derived an exact condition on the artificial boundary $\mathscr{B}$ in the Laplace–Fourier space, and then localized this condition in space, to obtain a NRBC which is nonlocal in time alone. A finite difference scheme was used in Ω. In [118] Gustafsson applied this technique to the scalar wave equation, by first [illegible] stability of the scheme.

Ting and Miksis [119] proposed an exact nonlocal boundary condition for three-dimensional problems in acoustics, using *two* artificial boundaries. Their setup is shown in Fig. 6. The computational domain Ω is bounded internally by the surface of the scatterer Γ, and externally by the artificial boundary $\mathscr{B}$. The second artificial boundary, $\mathscr{S}$, is a number of grid points away, inside Ω. The authors made use of the Kirchhoff formula for the reflected wave:

$$u(\mathbf{x}, t) = -\frac{1}{4\pi}\int_{\mathscr{S}}\left([u]\frac{\partial}{\partial n}\left(\frac{1}{r}\right) - \frac{1}{r}\left[\frac{\partial u}{\partial n}\right] - \frac{1}{rc}\frac{\partial r}{\partial n}\left[\frac{\partial u}{\partial t}\right]\right) d\mathbf{x}_{\mathscr{S}}. \tag{27}$$

Here $\mathbf{x}$ is a point outside $\mathscr{S}$, $\mathbf{x}_{\mathscr{S}}$ is a point on $\mathscr{S}$, $r = |\mathbf{x} - \mathbf{x}_{\mathscr{S}}|$, and $\partial/\partial n$ is the normal derivative at $\mathbf{x}_{\mathscr{S}}$. The operator $[\cdot]$ is the retarded value operator, namely

$$[f] = f(\mathbf{x}_{\mathscr{S}}, \tau)|_{\tau = t - r/c}. \tag{28}$$

Now choosing $\mathbf{x}$ to lie on the boundary $\mathscr{B}$, (27) can be used as an exact boundary condition, which involves u on $\mathscr{B}$ and the retarded values of u, $\partial u/\partial n$, and $\partial u/\partial t$ on $\mathscr{S}$. A similar boundary condition is proposed for the reduced wave equation.

The exact NRBC (27) is nonlocal in space and time. The nonlocality in time is, however, limited to a fixed amount of required past data, because the retarded value $\tau = t - r/c$ is bounded by $t - r_{\max}/c \leqslant \tau \leqslant t - r_{\min}/c$, where $r_{\min}$ and $r_{\max}$ are the extremal values of r. In other words, the memory required by the numerical scheme does not grow in time. Clearly, this is the big advantage of the scheme. It is interesting to note that the NRBC (27) is inherently three-dimensional. In a sense, the two-dimensional case is harder, because an exact boundary condition in that case would not have a limited nonlocality in time such as in (27). This is related to the fact that the Green's function in even dimensions has an infinitely long "tail." Ting and Miksis left the implementation and the testing of their scheme to other researchers, a work still to be done.

An exact NRBC for time-harmonic problems, of the form

$$\frac{\partial u}{\partial n} = Mu \qquad \text{on } \mathscr{B}, \tag{29}$$

was proposed by Fix and Marin [120], MacCamy and Marin [121], Marin [122], Goldstein [123], Lenoir and Tounsi [124], Canuto *et al.* [125], Keller and Givoli

[126], and Givoli and Keller [127, 128]. In (29), M is a nonlocal operator called the *Dirichlet to Neumann* map, because it relates the Dirichlet datum u to the Neumann datum $\partial u/\partial n$. In all of these works, except the one by Canuto *et al.*, the finite element method is employed in the computational domain. This is not accidental; the NRBC (29) is highly compatible with the finite element method. In [120] this boundary condition is found analytically for the reduced wave equation in a waveguide. In [121] two-dimensional exterior domains are considered. The boundary condition involves the solution of an integral equation on $\mathscr{B}$ for which numerical methods of solution are given. The convergence of the finite element method with this boundary condition is proved. A summary of these results and some numerical examples are presented in [122].

Goldstein [123] derives a NRBC of the form (29) using the eigenvalues and eigenfunctions of the Laplacian, and gives a long and very technical convergence proof for the application of the scheme in waveguides. He shows that in this case the reflected waves contain a finite number of propagation modes, whereas the solution corresponding to all the higher modes is exponentially decaying. Thus, the infinite series in the exact boundary condition can be truncated to include only the propagating modes. In exterior problems, on the other hand, there is an infinite number of propagation modes, therefore more terms are typically needed to be taken into account in the NRBC. Bayliss *et al.* [129] used Goldstein's NRBC for wave guides, employing a finite element scheme with a preconditioned conjugate gradient linear equation solver.

Lenoir and Tounsi [124] also used a NRBC of the form (29) in solving the problem of the potential flow around a ship. The geometry that was considered is that of an infinite wave guide, where Laplace's equation governs. The convergence of the scheme was discussed in detail.

Canuto *et al.* [125] proposed a similar procedure for exterior problems in two dimensions. However, they incorporated the NRBC (29) in a spectral scheme, as opposed to all the previously mentioned works which employed the finite element method. To this end, the authors first transformed the computational domain into a rectangular domain. The resulting NRBC is called the Infinite Order Radiation Condition. The delightfully clear review paper by Hariharan [130] summarizes this method and compares it with that of Bayliss *et al.* [28]. In addition, this review includes a detailed discussion on the nonlocal NRBC of MacCamy and Marin [121].

Keller and Givoli [126] obtained an explicit expression for the exact Dirichlet to Neumann boundary condition (29) in two- and three-dimensional exterior problems. In order to do this, the boundary $\mathscr{B}$ was chosen to be a circle in two dimensions and a sphere in three dimensions. For the two-dimensional reduced wave equation (1), on a circular boundary $\mathscr{B}$ of radius R, the exact boundary condition is

$$\frac{\partial u}{\partial n}(R, \theta) = -\sum_{n=0}^{\infty}{}' \int_0^{2\pi} m_n(\theta - \theta')\, u(R, \theta')\, d\theta', \tag{30}$$

where

$$m_n(\theta-\theta') = -\frac{k}{\pi}\frac{H_n^{(1)\prime}(kR)}{H_n^{(1)}(kR)}\cos n(\theta-\theta'). \tag{31}$$

Here $H_n^{(1)}$ is the Hankel function of the first kind. The prime after the sum in (30) indicates that a factor of $\frac{1}{2}$ multiplies the term with $n=0$.

It is shown in [126] that the method of combining the exact boundary condition (30) with the finite element method is very effective. At first sight it seems that the nonlocality of the condition (30) might spoil the banded structure of the finite element matrix, and the complexity of this condition might require a great deal of computation. However, neither of these difficulties occurs. In fact, the results presented in [126] are more accurate than those obtained by using the approximate local conditions (14) and (17), while requiring about the same amount of computational work.

In Givoli and Keller [127], an exact closed-form boundary condition of the Dirichlet to Neumann type for elastic waves is obtained and combined with a finite element scheme in Ω. The numerical examples demonstrate the superiority of the method over some of the local NRBCs for time-harmonic elastodynamics. The Dirichlet to Neumann finite element method is generalized and adapted to a variety of other problems in Givoli and Keller [128]. A work currently under way is concerned with the generalization of the NRBC (29) to the time-dependent case.

6. Concluding Remarks

The interest in NRBCs has started in the early 1970s, and grew constantly since then. Yet much work is still to be done. Future work will include the adaptation and application of existing NRBCs to situations more complicated than those for which they had been originally devised. This will involve the extension of two-dimensional NRBCs to three dimensions, extension of steady-state NRBCs to the time-dependent case, and application of NRBCs designed for linear problems in the nonlinear regime.

Only a few works have dealt with nonlinear problems in which the computational domain and the exterior domain interact. Much research on this issue is still ahead. In fact, any kind of NRBC represents (implicitly or otherwise) the physical and mathematical behavior of the solution at the exterior domain (and at infinity). In many cases a poor performance of a NRBC originates from inappropriate modeling of the exterior. This is related to the violation of point number 2 in the "goal list" of Section 1.

As we have seen in the previous sections, local NRBCs have been constructed in many cases so that spurious reflections are kept very small (or vanish entirely) in certain "modes," or at certain angles of incidence, or for a certain range of frequencies, or in a certain sense of average. Therefore many local NRBCs perform very well in some situations and quite poorly in others. This is why we feel that exact nonlocal boundary conditions are more promising. Their robustness and accuracy have the potential of overshadowing the apparent simplicity of using local boundary conditions. Of course, every new nonlocal NRBC has to pass the test of computational efficiency. This is, we believe, an important direction in future research on NRBCs.

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